Principles Of Mathematical Induction

1. Give an example of a statement P(n) which is true for all $n \geq 4$ but P(1), P(2) and P(3) are not true. Justify your answer.
2. Give an example of a statement P(n) which is true for all n. Justify your answer.

Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction :

3. $4^n - 1$ is divisible by 3, for each natural number n.
4. $2^{3n} - 1$ is divisible by 7, for all natural numbers n.
5. $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.
6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n.
7. For any natural number n, $7^n - 2^n$ is divisible by 5.
8. For any natural number n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.
9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.
10. $n(n^2 + 5)$ is divisible by 6, for each natural number n.
11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.
12. $2n < (n + 2)!$ for all natural number n.
13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.
14. $2 + 4 + 6 + \ldots + 2n = n^2 + n$ for all natural numbers n.
15. $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all natural numbers n.
16. $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$ for all natural numbers n.

Use the Principle of Mathematical Induction in the following Exercises.

17. A sequence $a_1, a_2, a_3 \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all natural numbers.

18. A sequence $b_0, b_1, b_2 \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k. Show that $b_n = 5 + 4n$ for all natural number n using mathematical induction.

19. A sequence $d_1, d_2, d_3 \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in \mathbf{N}$.

20. Prove that for all $n \in \mathbf{N}$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$$

$$= \frac{\cos\left(\alpha+\left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

21. Prove that, $\cos\theta\cos 2\theta\cos 2^2\theta \ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}$, for all $n \in \mathbf{N}$.

22. Prove that, $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$, for all $n \in \mathbf{N}$.

23. Show that $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number for all $n \in \mathbf{N}$.

24. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n > 1$.

25. Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in \mathbf{N}$.

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

26. If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in \mathbf{N}$, then the least positive integral value of k is

(A) 5 (B) 3 (C) 7 (D) 1

27. For all $n \in \mathbf{N}$, $3.5^{2n+1} + 2^{3n+1}$ is divisible by

(A) 19 (B) 17 (C) 23 (D) 25

28. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(A) 1 (B) 2 (C) 3 (D) 4